

A NEW CHARACTERIZATION OF BAIRE CLASS 1 FUNCTIONS

LUCA MOTTO ROS

ABSTRACT. We give a new characterization of the Baire class 1 functions (defined on an ultrametric space) by proving that they are exactly the pointwise limits of sequences of full functions (which are particularly simple Lipschitz functions). Moreover we highlight the link between the two classical stratifications of the Borel functions by showing that the Baire class functions of some level are exactly those obtained as uniform limits of sequences of Delta functions (of a corresponding level).

1. INTRODUCTION

If X and Y are metrizable spaces, a function $f : X \rightarrow Y$ is said to be *continuous* if the preimage of an open set of Y is open with respect to the topology of X , i.e. if $f^{-1}(U) \in \Sigma_1^0(X)$ for every $U \in \Sigma_1^0(Y)$. There are two natural generalizations of this definition, namely functions such that $f^{-1}(U) \in \Sigma_{\xi+1}^0(X)$ for every U open in Y and functions such that $f^{-1}(S) \in \Sigma_\xi^0(X)$ for every $S \in \Sigma_\xi^0(Y)$ (for $\xi < \omega_1$): the former are called *Baire class functions (of level ξ)* while the latter are called *Delta functions (of level ξ)*. Each generalization provides a stratification of the Borel functions from X to Y , but if we compare the levels of the two hierarchies, that is if we fix some $\xi < \omega_1$ in the definitions above, they are quite different: for example, each level of the Delta functions is closed under composition, while no level of the Baire class functions (apart from continuous functions) has such a property.

The Baire class stratification was introduced by Baire in 1899 (with a slightly different definition which, however, turns out to be equivalent to the one proposed here in the relevant cases) and has been extensively studied. Of particular interest are the Baire class 1 functions, i.e. those functions such that the preimage of an open set is a Σ_2^0 set. For example, if $f : [0, 1] \rightarrow \mathbb{R}$ is differentiable (at endpoints we take one-side derivatives), then its derivative f' is of Baire class 1. Moreover, Baire class 1 functions (in particular those from the Baire space ${}^\omega\omega$ or from any compact space X to \mathbb{R}) have lots of applications in the theory of Banach spaces (for more on this subject see, for example, [4], [2], [3], [6], [5] and references quoted there).

In this paper we will give a new characterization for the Baire class 1 functions defined from an ultrametric space X (such as the Baire space ${}^\omega\omega$ or the Cantor space ${}^\omega 2$) to any separable metric space Y , by showing that they are exactly the pointwise limits of sequences of full functions (which are particular Lipschitz functions) between X and Y . Moreover we will show that the two hierarchies presented

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before are intimately related by proving that a function is of level ξ in the Baire class stratification just in case it is the uniform limit of functions of level $\xi + 1$ in the Delta stratification. In particular, this gives another characterization of the Baire class 1 functions (taking $\xi = 1$).

The paper is organized as follows. In Section 2 we give some (old and new) definitions and state the main Theorems of the paper. In Section 3 we consider the relations between Baire class and Delta functions, while in Section 4 we prove some Theorems about zero-dimensional and ultrametric spaces. The results of these two Sections are partially implicit in some classical proofs, but we put them here since we want to highlight the link between the two stratifications of the Borel-functions and the special properties of Borel-partitions of completely disconnected spaces. Finally, in Section 5 we give the proof of the new characterization of the Baire class 1 functions.

All the proofs need only a very small fragment of the Axiom of Choice, namely Countable Choice over the Reals ($\text{AC}_\omega(\mathbb{R})$ for short)¹. It seems not possible to avoid this (very weak) assumption since it is needed even to prove very basic results in Descriptive Set Theory, e.g. to prove that $\Sigma_2^0(\mathbb{R})$ is closed under countable unions. Hence we will always work under $\text{ZF} + \text{AC}_\omega(\mathbb{R})$. All the metrics d considered throughout the paper are always assumed to be such that $d \leq 1$. This condition is needed for the proofs of some of the results, but it is not a true limitation. In fact, given any metric d on X , it is easy to see that $d' = \frac{d}{1+d}$ is a metric on X compatible with d such that $d' \leq 1$. Moreover, d is an ultrametric if and only if d' is an ultrametric, and one can easily check that all the definitions given in this paper are “invariant” under such a transformation of the metric, e.g. $A \subseteq X$ is a full set with respect to d just in case it is a full set with respect to d' (although with different constants²). Thus all the results hold also when considering *arbitrary* (ultra)metrics. Finally, given any two sets A and B , we will denote by ${}^A B$ the set of all the functions from A to B and by ${}^{<\omega} A$ the set of all the *finite* sequences of elements from A . In particular, ${}^\omega \omega$ (the set of all the ω -sequences of natural numbers) will denote the Baire space (endowed with the usual topology), while ${}^{<\omega} \omega$ will denote the set of all the *finite* sequences of natural numbers. For all the other undefined concepts and symbols we will always refer the reader to the standard monograph [1].

Finally, it is the author’s pleasure to acknowledge his debt to Sławomir Solecki for his review of the present work and for the suggestion of a further generalization of the characterization previously obtained.

2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

We start with a few of definitions and basic results, following closely the presentation of [1].

¹The fact that we will not use the full Axiom of Choice becomes relevant if one wants to assume other axioms which contradict AC (which however are, in general, consistent with $\text{AC}_\omega(\mathbb{R})$). For example, the Axiom of Determinacy AD is needed to carry out the Wadge’s analysis of continuous reducibility, so it could be useful to check that our results hold also in that context.

²In particular, one constant can be obtained from the other one via the bijection $j : \mathbb{R}^+ \rightarrow (0, 1) : r \mapsto \frac{r}{1+r}$.

Definition 1. Let X, Y be metrizable spaces and $\xi < \omega_1$ a nonzero ordinal. A function $f : X \rightarrow Y$ is of *Baire class 1* if $f^{-1}(U) \in \Sigma_2^0(X)$ for every open set $U \subseteq Y$. Recursively, for $1 < \xi < \omega_1$ we define now a function $f : X \rightarrow Y$ to be of *Baire class ξ* if it is the pointwise limit of a sequence of functions $f_n : X \rightarrow Y$, where f_n is of Baire class $\xi_n < \xi$.

We denote by $\mathcal{B}_\xi(X, Y)$ the set of Baire class ξ functions from X into Y .

A function f which is of Baire class ξ (for some nonzero countable ordinal ξ) is called a *Baire class function*.

Definition 2. Let X, Y be metrizable spaces and let Γ be some collection of subsets of X . We say that $f : X \rightarrow Y$ is Γ -*measurable* if $f^{-1}(U) \in \Gamma$ for every open set $U \subseteq Y$.

The link between Γ -measurable and Baire class function is given by the following classical Theorem.

Theorem 2.1 (Lebesgue, Hausdorff, Banach). *Let X, Y be metrizable spaces, with Y separable. Then for $1 \leq \xi < \omega_1$, $f : X \rightarrow Y$ is of Baire class ξ if and only if f is $\Sigma_{\xi+1}^0$ -measurable.*

By analogy with respect to this Theorem, we say that a function f between two metrizable spaces is of *Baire class 0* if and only if it is Σ_1^0 -measurable, i.e. if and only if f is continuous.

As a consequence of this Theorem, if X and Y are metrizable spaces and Y is separable, then the Baire class ξ functions provide a stratification in ω_1 levels of all the Borel functions, i.e. functions such that $f^{-1}(U)$ is Borel for any $U \in \Sigma_1^0(Y)$ (Borel-measurable functions). In fact for every nonzero countable ξ and every $f \in \mathcal{B}_\xi(X, Y)$, f is clearly Borel. Conversely, let U_n be a countable basis for the topology of Y and let f be Borel. Let μ_n be nonzero countable ordinals such that $f^{-1}(U_n) \in \Sigma_{\mu_n}^0$ and let $\xi = \sup\{\mu_n \mid n \in \omega\}$ (which is again a nonzero countable ordinal). Since Σ_ξ^0 is closed under countable unions and $f^{-1}(U_n) \in \Sigma_\xi^0$ for every $n \in \omega$, we have that $f \in \mathcal{B}_\xi(X, Y)$. Note also that any $\mathcal{B}_\xi(X, X)$ is not closed under composition since, in general, if $f \in \mathcal{B}_\mu(X, Y)$ and $g \in \mathcal{B}_\nu(Y, Z)$ then $g \circ f \in \mathcal{B}_{\mu+\nu}(X, Z)$. This result follows from the fact that if $A \in \Sigma_\nu^0(Y)$ and $f \in \mathcal{B}_\mu(X, Y)$ then $f^{-1}(A) \in \Sigma_{\mu+\nu}^0$.

The following is another classical fact³.

Theorem 2.2 (Lebesgue, Hausdorff, Banach). *Let X, Y be separable metrizable. Moreover, assume that either X is zero-dimensional or $Y = \mathbb{R}^n$ for some $n \in \omega$ (or even $Y = \mathbb{C}^m$ or $Y = [0, 1]^m$ for some $m \in \omega$). Then $f : X \rightarrow Y$ is of Baire class 1 if and only if f is the pointwise limit of a sequence of continuous functions.*

Hence, under the hypotheses of this Theorem, $f \in \mathcal{B}_\xi(X, Y)$ if and only if it is the pointwise limit of a sequence of functions in $\bigcup_{\nu < \xi} \mathcal{B}_\nu(X, Y)$, for all $\xi \geq 1$.

³In general, if X and Y are metrizable with Y separable and $f : X \rightarrow Y$ is the pointwise limit of a sequence of continuous functions then f is of Baire class 1. Nevertheless the converse fails in the general case: for a counterexample, simply take $X = \mathbb{R}$ and $Y = \{0, 1\}$ (with the discrete metric) and consider the function such that $f(0) = 1$ and $f(x) = 0$ for every $x \neq 0$.

There is another stratification of the Borel functions (in the case Y separable) which is important because, contrary to the case of Baire class functions, every level is a set of functions closed under composition.

Definition 3. Let X, Y be metrizable spaces and $\xi < \omega_1$ be a nonzero ordinal. A function $f : X \rightarrow Y$ is a Δ_ξ^0 -function (Δ_ξ^0 for short) if $f^{-1}(A) \in \Sigma_\xi^0(X)$ for every $A \in \Sigma_\xi^0(Y)$.

We denote by $\mathcal{D}_\xi(X, Y)$ the set of such functions.

Proposition 2.3. For $\xi > 1$ the following are equivalent⁴:

- i) f is Δ_ξ^0 ;
- ii) $f^{-1}(A) \in \Pi_\xi^0$ for every $A \in \Pi_\xi^0$;
- iii) $f^{-1}(A) \in \Delta_\xi^0$ for every $A \in \Delta_\xi^0$;
- iv) $f^{-1}(A) \in \Sigma_\xi^0$ for every $\nu < \xi$ and $A \in \Pi_\nu^0$;
- v) $f^{-1}(A) \in \Delta_\xi^0$ for every $\nu < \xi$ and $A \in \Sigma_\nu^0$.

Proof. Since Σ_ξ^0 is closed under countable union, it is easy to see that $i) \iff iii)$. $i) \iff ii)$ is obvious, and also $iii) \Rightarrow v)$ is trivial (since $\Sigma_\nu^0 \subseteq \Delta_\xi^0$ for every $\nu < \xi$). $v) \Rightarrow iv)$ since Δ_ξ^0 is closed under complementation and is contained by definition in Σ_ξ^0 . Finally, to see that $iv) \Rightarrow i)$ recall that, by definition, every Σ_ξ^0 set A can be written as a countable union of $\bigcup_{\nu < \xi} \Pi_\nu^0$ sets. \square

As in the case of Baire class functions, a function f which is a Δ_ξ^0 -function (for some nonzero countable ordinal ξ) is called a *Delta function*.

To observe that the Delta functions provide a stratification in ω_1 levels of all the Borel functions it is enough to observe that every open set of Y is in $\Sigma_\xi^0(Y)$ for every nonzero countable ordinal ξ and every metrizable space Y (and hence every Delta function is Borel) and that every Baire class function is a Delta function. To see this, let $f \in \mathcal{B}_\nu(X, Y)$ and let ξ be the first additively closed ordinal above ν (that is $\xi = \nu \cdot \omega$): we claim that f is a Δ_ξ^0 -function. In fact, let $S \in \Sigma_\xi^0$: by definition, $S = \bigcup_n P_n$, where each $P_n \in \Pi_{\mu_n}^0(Y)$ for some $\mu_n < \xi$. Since $f \in \mathcal{B}_\nu(X, Y)$ we have that $Q_n = f^{-1}(P_n) \in \Pi_{\nu+\mu_n}(X)$ and hence $f^{-1}(S) = \bigcup_n Q_n$ where each Q_n is in $\Pi_{\nu+\mu_n}^0(X)$. Since ξ is additively closed and $\nu, \mu_n < \xi$ we have that $\nu + \mu_n < \xi$ for every $n \in \omega$: therefore $f^{-1}(S) \in \Sigma_\xi^0(X)$ by definition.

Moreover, using again the fact that $\Sigma_1^0(Y) \subseteq \Sigma_\xi^0(Y)$, it is easy to check that $\mathcal{D}_{\xi+1}(X, Y) \subseteq \mathcal{B}_\xi(X, Y)$.

Definition 4. Let X and Y be two metrizable spaces and let $\mathcal{F} \subseteq \mathcal{G}$ be two sets of functions from X to Y . Then \mathcal{F} is a *basis* for \mathcal{G} just in case every function in \mathcal{G} is the *uniform* limit of a sequence of functions in \mathcal{F} .

We will prove in Section 3 that each level of the Delta functions forms a basis for a corresponding level of the Baire class functions. This result is essentially implicit in the proof of Theorem 2.1 (see [1]), but we will reprove it here for the sake of completeness.

⁴For $\xi = 1$ we have in general that $i) \iff ii) \Rightarrow iii) \iff iv) \iff v)$ but not $iii) \Rightarrow i)$ (in fact if Y is connected we have that *every* function f satisfies $iii)$, but f is a Δ_1^0 -function if and only if f is continuous). Nevertheless the Proposition remains true even for $\xi = 1$ if we require that Y is zero-dimensional.

Theorem 2.4. *Let (X, d_X) , (Y, d_Y) be two metric spaces and assume that Y is also separable. A function $f : X \rightarrow Y$ is of Baire class ξ if and only if it is the uniform limit of a sequence of $\Delta_{\xi+1}^0$ -functions.*

Corollary 2.5. *Let (X, d_X) , (Y, d_Y) be two metric spaces and assume that Y is also separable. A function $f : X \rightarrow Y$ is in $\mathcal{B}_1(X, Y)$ if and only if it is the uniform limit of a sequence of Δ_2^0 -functions.*

From Theorem 2.4 we can also derive the following Corollary. It can be seen as an extension of Theorem 2.2: in that case it was proved (under stronger hypotheses) that f is of Baire class 1 if and only if it is the pointwise limit of a sequence of Δ_1^0 -functions (i.e. continuous functions). Here we prove the same result for every level different from 1 (under weaker hypotheses).

Corollary 2.6. *Let X, Y be two metrizable spaces and assume that Y is also separable. Then for every $1 < \xi < \omega_1$, $f : X \rightarrow Y$ is of Baire class ξ if and only if f is the pointwise limit of a sequence of Δ_ξ^0 -functions.*

By Theorem 2.2, as previously observed, Corollary 2.6 remains true in the case $\xi = 1$ if we require that X is separable and either X is zero-dimensional or Y is one of \mathbb{R}^n , $[0, 1]^n$ or \mathbb{C}^n (for some $n \in \omega$).

Finally, we want to give a new characterization of the Baire class 1 functions. First recall the following Definition.

Definition 5. Let (X, d_X) , (Y, d_Y) be two metric spaces. A function $f : X \rightarrow Y$ is *Lipschitz* (with constant $L \in \mathbb{R}^+$) if

$$\forall x, x' \in X (d_Y(f(x), f(x')) \leq L \cdot d_X(x, x')).$$

We denote by $\text{Lip}(X, Y; L)$ the set of such functions and put $\text{Lip}(X, Y) = \bigcup_{L \in \mathbb{R}^+} \text{Lip}(X, Y; L)$.

Let now (X, d_X) be an *ultrametric space*, i.e. a metric space such that d_X is an ultrametric. A set $A \subseteq X$ is *full* (with constant $r \in \mathbb{R}^+$) if

$$\forall x \in A (B(x, r) \subseteq A),$$

where $B(x, r) = \{y \in X \mid d_X(x, y) < r\}$ is the usual open ball.

Proposition 2.7. *Let (X, d_X) be an ultrametric space. Then the full subsets of X form an algebra. Moreover, an arbitrary union of balls with a fixed radius is full (in particular, an arbitrary union of full sets with the same constant is full).*

Proof. Let A and B be full sets with constants r_A and r_B respectively. Then it is easy to check that $A \cup B$ is full with constant $r = \min\{r_A, r_B\}$. Moreover, let $x \notin A$ and assume towards a contradiction that $y \in A$ for some $y \in B(x, r_A)$. By the properties of the ultrametric d_X , we have that $B(y, r_A) = B(x, r_A)$: but since A is full with constant r_A , then $B(y, r_A) \subseteq A$ and hence $x \in A$, a contradiction! Thus $X \setminus A$ is full (with constant r_A). The second part follows again from the properties of an ultrametric. \square

Definition 6. Let (X, d_X) be an ultrametric space and Y be any separable metrizable space. A function $f : X \rightarrow Y$ is said to be *full* if it has only finitely many values and the preimage of each of these values is a full set.

The function f is said to be ω -full if it has at most countably many values and there is some fixed $r \in \mathbb{R}^+$ such that the preimage of each value is a full set with constant r .

It is clear that every full function is ω -full. Moreover, if f is ω -full and $r \in \mathbb{R}^+$ witnesses this, then $f \in \text{Lip}(X, Y; r^{-1})$ (with respect to any metric d_Y compatible with the topology of Y such that $d_Y \leq 1$). In fact, let d_Y be such a metric and let $x, x' \in X$: if $d_X(x, x') \geq r$ then

$$d_Y(f(x), f(x')) \leq 1 = r^{-1} \cdot r \leq r^{-1} d_X(x, x'),$$

while if $d_X(x, x') < r$ then $x' \in f^{-1}(f(x))$ (since $x' \in B(x, r)$) and hence $f(x) = f(x')$.

Proposition 2.8. *Let (X, d_X) be an ultrametric space and (Y, d_Y) , (Z, d_Z) be two metric spaces. Let $f : X \rightarrow Y$ be a full function, $g \in \text{Lip}(Y, Z; L)$ and $h \in \text{Lip}(Z, X; L)$. Then $g \circ f$ is full and, if d_Z is an ultrametric, also $f \circ h$ is full.*

The same result holds if we systematically replace “full” with “ ω -full”.

Proof. The first part is obvious, since for every $z \in Z$ the set $(g \circ f)^{-1}(z)$ is either empty or the union of finitely many full sets (and the cardinality of $\text{range}(g \circ f)$ is less or equal than the cardinality of $\text{range}(f)$). For the second part, it is enough to show that the preimage via h of a full set $A \subseteq X$ (with constant r) is a full set (with constant $r \cdot L^{-1}$). In fact, let $z \in Z$ be such that $h(z) \in A$ and let $z' \in Z$ be such that $d_Z(z, z') < rL^{-1}$. Then $d_X(h(z), h(z')) \leq L \cdot d_Z(z, z') < LrL^{-1} = r$, and thus $h(z') \in A$. But this implies $B(z, rL^{-1}) \subseteq h^{-1}(A)$ and hence we are done. The case in which f is ω -full is proved in a similar way. \square

Now we are ready to state the main Theorem of this paper.

Theorem 2.9. *Let (X, d_X) be an ultrametric space and let Y be any separable metrizable space. Then $f : X \rightarrow Y$ is of Baire class 1 if and only if f is the pointwise limit of a sequence of full functions.*

By the observations above and since every Lipschitz function is uniformly continuous, we have also the following Corollary as a simple consequence of Theorem 2.9.

Corollary 2.10. *Let (X, d_X) , (Y, d_Y) be separable metric spaces and assume that X is an ultrametric space with respect to d_X . For every $f : X \rightarrow Y$ the following are equivalent:*

- i) f is of Baire class 1;
- ii) f is the pointwise limit of a sequence of ω -full functions;
- iii) f is the pointwise limit of a sequence of Lipschitz functions;
- iv) f is the pointwise limit of a sequence of uniformly continuous functions.

The author first proved Theorem 2.9 but using Lipschitz (in particular ω -full) functions rather than full functions (although the proof was essentially the same presented here in Section 5): the idea to generalize the result to the present form (as well as the definition of fullness) is due to S. Solecki.

3. THE LINK BETWEEN BAIRE CLASS AND DELTA FUNCTIONS

We first give some basic definitions.

Definition 7. Let X be a topological space and $\Gamma \subseteq \mathcal{P}(X)$ be any pointclass. A Γ -partition of a set $C \in \Gamma$ is a family $\langle C_n \mid n < N \rangle$ of nonempty pairwise disjoint sets of Γ such that $C = \bigcup_{n < N} C_n$ and $1 \leq N \leq \omega$ ⁵.

Definition 8. Let X, Y be two metrizable spaces and let \mathcal{F} be some set of functions between X and Y . Let $f : X \rightarrow Y$ be an arbitrary function and $\langle C_n \mid n < N \rangle$ be some partition of X . We say that f is *(locally) in \mathcal{F} on the partition $\langle C_n \mid n < N \rangle$* if there is a family of functions $\{f_n \mid n < N\} \subseteq \mathcal{F}$ such that $f \upharpoonright C_n = f_n \upharpoonright C_n$ for every $n < N$.

Moreover, if $\Gamma \subseteq \mathcal{P}(X)$ is any pointclass, we will say that f is *(locally) in \mathcal{F} on a Γ -partition* if there is some Γ -partition such that f is locally in \mathcal{F} on it.

Obviously, if \mathcal{F} and \mathcal{G} are sets of functions and $\mathcal{F} \subseteq \mathcal{G}$, then if f is locally in \mathcal{F} on the partition $\langle C_n \mid n < N \rangle$ we have also that f is locally in \mathcal{G} on the same partition.

Proposition 3.1. *Let X, Y be two metrizable spaces and ξ be some nonzero countable ordinal. Then every $f : X \rightarrow Y$ is in $\mathcal{D}_\xi(X, Y)$ if and only if there is a Σ_ξ^0 -partition of X such that f is locally in $\mathcal{D}_\xi(X, Y)$ on it.*

Proof. One direction is trivial, hence we have only to prove that if $\langle C_n \mid n < N \rangle$ is a $\Sigma_\xi^0(X)$ -partition on X and $\{f_n \mid n < N\} \subseteq \mathcal{D}_\xi(X, Y)$ witnesses that f is locally in $\mathcal{D}_\xi(X, Y)$ on it, then $f \in \mathcal{D}_\xi(X, Y)$. To see this, let $S \in \Sigma_\xi^0(Y)$. Then $f^{-1}(S) = \bigcup_{n < N} (f^{-1}(S) \cap C_n) = \bigcup_{n < N} (f_n^{-1}(S) \cap C_n)$: but $f_n^{-1}(S) \cap C_n \in \Sigma_\xi^0(X)$ for every $n < N$ and therefore $f^{-1}(S) \in \Sigma_\xi^0(X)$ (since $\Sigma_\xi^0(X)$ is closed under countable unions). \square

We are now ready to prove a Theorem from which Theorem 2.4 easily follows. We will use the following standard fact.

Fact 1. Let $(X, d_X), (Y, d_Y)$ be two metric spaces, ξ be a nonzero countable ordinal and let $\langle f_n \mid n \in \omega \rangle$ be a sequence of functions from $\mathcal{B}_\xi(X, Y)$ converging uniformly to some $f : X \rightarrow Y$. Then $f \in \mathcal{B}_\xi(X, Y)$ as well.

Theorem 3.2. *Let $(X, d_X), (Y, d_Y)$ be two metric spaces and assume that Y is also separable. Then for every function $f : X \rightarrow Y$ the following conditions are equivalent (for⁶ $1 \leq \xi < \omega_1$):*

- i) f is of Baire class ξ ;
- ii) there is a sequence of functions $\langle f_k \mid k \in \omega \rangle$ which converges uniformly to f and such that every f_k is locally constant on some $\Sigma_{\xi+1}^0$ -partition;
- iii) there is a sequence of functions $\langle f_k \mid k \in \omega \rangle$ which converges uniformly to f and such that every f_k is locally Lipschitz on some $\Sigma_{\xi+1}^0$ -partition;
- iv) there is a sequence of functions $\langle f_k \mid k \in \omega \rangle$ which converges uniformly to f and such that every f_k is locally continuous on some $\Sigma_{\xi+1}^0$ -partition;
- v) there is a sequence of functions $\langle f_k \mid k \in \omega \rangle$ which converges uniformly to f and such that every f_k is a $\Delta_{\xi+1}^0$ -function.

⁵For the rest of the paper we will always assume without explicitly mentioning it that N is some ordinal less or equal to ω .

⁶If $\xi = 0$ then it is trivially true that i) \iff iv) \iff v), but ii) and iii) are not equivalent to i) unless X is zero-dimensional.

Proof. It is obvious that $ii) \Rightarrow iii)$ and $iii) \Rightarrow iv)$, since every constant function is Lipschitz and every Lipschitz function is also continuous. Moreover, using Proposition 3.1 and the fact that every continuous function is Δ_ξ^0 (for every nonzero $\xi < \omega_1$) we have also $iv) \Rightarrow v)$. Also $v) \Rightarrow i)$ is easy: in fact every $\Delta_{\xi+1}^0$ -function is of Baire class ξ and $\mathcal{B}_\xi(X, Y)$ is closed under uniform limits by the Fact above.

Finally we prove $i) \Rightarrow ii)$. For every $k \in \omega$, fix some open cover $\langle U_n^k \mid n < N \rangle$ of Y of mesh 2^{-k} , that is a sequence of open sets such that $Y = \bigcup_{n < N} U_n^k$ and $\text{diam}(U_n^k) \leq 2^{-k}$ for every $n < N$, and fix also a sequence $\langle z_n^k \mid n < N \rangle$ of points of Y such that $z_n^k \in U_n^k$ for every $n < N$. One way to do this is to consider an enumeration $\langle y_n \mid n < N \rangle$ of some (at most) countable dense set of points of Y (which exists because Y is separable) and to put $U_n^k = B(y_n, 2^{-(k+1)}) = \{y \in Y \mid d_Y(y, y_n) < 2^{-(k+1)}\}$ and $z_n^k = y_n$. Let $S_n^k = f^{-1}(U_n^k)$. Clearly $\forall n < N (S_n^k \in \Sigma_{\xi+1}^0(X))$ (since $f \in \mathcal{B}_\xi(X, Y)$) and the sets $\langle S_n^k \mid n < N \rangle$ cover X . Since $\Sigma_{\xi+1}^0$ has the generalized reduction property, we can find for every $k \in \omega$ a sequence $\langle Q_n^k \mid n < N \rangle$ of $\Sigma_{\xi+1}^0$ sets such that $Q_n^k \subseteq S_n^k$ for every $n < N$, $Q_n^k \cap Q_m^k = \emptyset$ if $n \neq m$ and $\bigcup_{n < N} Q_n^k = \bigcup_{n < N} S_n^k = X$ (thus, in particular, the Q_n^k 's form a $\Sigma_{\xi+1}^0$ -partition of X).

Now define $f_k : X \rightarrow Y : x \mapsto z_n^k$, where $n < N$ is the unique natural number such that $x \in Q_n^k$.

Note that f_k is locally constant on $\langle Q_n^k \mid n < N \rangle$. It remains only to prove that the sequence $\langle f_k \mid k \in \omega \rangle$ converges uniformly to f . Clearly this follows from

Claim 3.2.1. Fix some $k \in \omega$. Then for every $x \in X$

$$d(f_k(x), f(x)) \leq 2^{-k}.$$

Proof of the Claim. Fix some $x \in X$ and let n be such that $x \in Q_n^k$ (so that, in particular, $x \in S_n^k$). Then $f(x) \in U_n^k$, and since $\text{diam}(U_n^k) \leq 2^{-k}$ and $z_n^k \in U_n^k$ we have that $d(f_k(x), f(x)) \leq 2^{-k}$. \square *Claim*

\square

Remark 3.3. The same result holds if we consider $\Delta_{\xi+1}^0$ -partitions because it is easy to check that every $\Sigma_{\xi+1}^0$ -partition is actually a $\Delta_{\xi+1}^0$ -partition (the converse is trivially true). We can also consider partitions formed only by sets which are difference of two Π_ξ^0 sets (i.e. $2\text{-}\Pi_\xi^0$ sets), since every $\Sigma_{\xi+1}^0$ -partition $\langle S_n \mid n < N \rangle$ can be refined to a $2\text{-}\Pi_\xi^0$ -partition. In fact, since $S_n \in \Sigma_{\xi+1}^0$, by definition there are $\langle P_{n,m} \mid m \in \omega, n < N \rangle$ such that $P_{n,m} \in \Pi_\xi^0$ and $S_n = \bigcup_{m \in \omega} P_{n,m}$ for every $n < N$ (we are not requiring that the sets $P_{n,m}$ are different for distinct indexes m , hence we can suppose that $P_{n,m}$ is defined for every $m \in \omega$). Fix some bijection $\langle \cdot, \cdot \rangle$ between $\omega \times \omega$ and ω (for example $\langle i, j \rangle = (2^i(2j+1)) - 1$) and let $R_{\langle n,m \rangle} = P_{n,m}$. Inductively put $j_0 = 0$ and $j_{i+1} = \min\{j \mid j > j_i \wedge R_j \setminus \bigcup_{l < j} R_l \neq \emptyset\}$ (in general the sequence j_i is defined for $i < I$ where $I \leq \omega$). Now define

$$Q_i = R_{j_i} \setminus \bigcup_{l < j_i} R_l = R_{j_i} \setminus \bigcup_{l < i} Q_l$$

for every $i < I$. Clearly $\langle Q_i \mid i < I \rangle$ is an at most countable partition of X (since the sets S_n cover X) and refines $\langle S_n \mid n < N \rangle$. Moreover every Q_i is the difference of two Π_ξ^0 sets (being Π_ξ^0 is closed under finite unions).

Moreover, if d_Y is a compact metric (e.g. it is induced by any metric on a compactification of Y), the partitions above can be taken to be finite.

Finally, as we will see in the next Section, if X is zero-dimensional we can strengthen the result a little bit by taking Π_ξ^0 -partitions (instead of $\Sigma_{\xi+1}^0$ -partitions) in conditions *ii*)-*v*).

Now we restate Corollary 2.6 in the following (slightly) stronger form.

Proposition 3.4. *Let X, Y be two metrizable spaces and assume that Y is also separable. Then for every nonzero $\xi < \omega_1$, if there is a sequence of Δ_ξ^0 -functions pointwise converging to f then f is of Baire class ξ .*

Conversely, if $\xi > 1$ and $f : X \rightarrow Y$ is of Baire class ξ then there is a sequence of Δ_ξ^0 -functions pointwise converging to f .

Proof. Let d be a compatible metric on Y . Recall that every open sphere U of Y can be written as the union of countably many closed sphere each of which is contained in the interior of the following one. In fact let $U = B(y_0, \varepsilon) = \{y \in Y \mid d(y, y_0) < \varepsilon\}$ and let $\langle \varepsilon_m \mid m \in \omega \rangle$ be a strictly increasing sequence of real such that $\varepsilon_m < \varepsilon$ for every $m \in \omega$ and $\lim_{m \in \omega} \varepsilon_m = \varepsilon$: then $U = \bigcup_{m \in \omega} \overline{B_m(y_0, \varepsilon_m)} = \bigcup_{m \in \omega} \{y \in Y \mid d(y, y_0) \leq \varepsilon_m\}$. Moreover, since $\varepsilon_n < \varepsilon_{n+1}$, we have also $\overline{B_m} = \overline{B(y_0, \varepsilon_m)} \subseteq B_{m+1} = B(y_0, \varepsilon_{m+1})$, that is $U = \bigcup_{m \in \omega} B_m$.

Now assume that $\langle f_k \mid k \in \omega \rangle$ is a sequence of Δ_ξ^0 -functions pointwise converging to f : it is enough to prove that $f^{-1}(U) \in \Sigma_{\xi+1}^0(X)$ for every open sphere $U = \bigcup_m \overline{B_m} \subseteq Y$. First note that

$$f^{-1}(U) = \bigcup_{m \in \omega} \bigcup_{n \in \omega} \bigcap_{k \geq n} f_k^{-1}(\overline{B_m}).$$

In fact, if $f(x) \in U$ then there is an m such that $f(x) \in B_m \subseteq \overline{B_m}$ and hence also $f_k(x) \in B_m$ for any k large enough (since f_k converge to f). For the other direction, if there is some m such that $f_k(x) \in \overline{B_m}$ for almost all k , thus also $f(x)$ (which is the limit of the points $f_k(x)$) must belong to the same $\overline{B_m}$ (since it is closed).

Since each f_k is a Δ_ξ^0 -function and since $\Pi_1^0(Y) \subseteq \Pi_\xi^0(Y)$ for every nonzero countable ξ , we have that $f^{-1}(\overline{B_m}) \in \Pi_\xi^0(X)$ for every $m \in \omega$ and hence also $\bigcap_{k \geq n} f_k^{-1}(\overline{B_m}) \in \Pi_\xi^0(X)$ for every $n \in \omega$ (since $\Pi_\xi^0(X)$ is closed under countable intersections). But then $f^{-1}(U)$ is a countable union of $\Pi_\xi^0(X)$ sets, i.e. it is a $\Sigma_{\xi+1}^0(X)$ set and we are done.

Conversely, if $\xi > 1$ and f is of Baire class ξ then it is the pointwise limit of some sequence f_n of functions such that for every $n \in \omega$ there is a $1 \leq \nu_n < \xi$ such that f_n is of Baire class ν_n . Using Theorem 2.4, find for every $n \in \omega$ a sequence $g_{n,m}$ of $\Delta_{\nu_n+1}^0$ -functions converging uniformly to f_n . Note that by the construction above (Claim 3.2.1) we can assume that $d(g_{n,m}(x), f_n(x)) \leq 2^{-m}$ for every $x \in X$. Moreover, since $\nu_n + 1 \leq \xi$ we have that every $g_{n,m}$ is, in particular, a Δ_ξ^0 -function. Take any diagonal subsequence $\langle h_n \mid n \in \omega \rangle$ of the $g_{n,m}$, e.g. $h_n = g_{n,n}$. It remains only to prove that this sequence converges pointwise to f . To see this, fix some $x \in X$ and $k \in \omega$. Let $j \in \omega$ be such that

$$\forall i \geq j \ (d(f_i(x), f(x)) < 2^{-(k+1)})$$

and put $m = \max\{j, k + 1\}$. Clearly, for every $m' \geq m$ we have

$$\begin{aligned} d(h_{m'}(x), f(x)) &\leq d(g_{m', m'}(x), f_{m'}(x)) + d(f_{m'}(x), f(x)) < \\ &< 2^{-m'} + 2^{-(k+1)} \leq 2 \cdot 2^{-(k+1)} = 2^{-k}. \end{aligned}$$

□

The same Corollary clearly holds if we consider functions which are constant (respectively, Lipschitz, continuous) on a *finite* Δ_ξ^0 -partition.

4. ZERO DIMENSIONAL SPACES

We now prove some Theorems on zero-dimensional and ultrametric spaces. In particular, the first is a simple variation of some classical results (see [1]). Let $s \in {}^{<\omega}\omega$ be a finite sequence of natural numbers. We will denote the *length* of s by $\text{lh}(s)$ (formally, $\text{lh}(s) = \text{dom}(s)$).

Theorem 4.1. *If (X, d) is a metric, separable and zero-dimensional space, then there is some set $A \subseteq {}^\omega\omega$ and an homeomorphism $h : A \rightarrow X$ such that $h \in \text{Lip}(A, X; 1)$ (with respect to d and the usual metric d' that ${}^\omega\omega$ induces on A).*

If moreover d is an ultrametric then h can be taken bi-Lipschitz, i.e. $h^{-1} \in \text{Lip}(X, A; 2)$ (and $h \in \text{Lip}(A, X; 1)$ as before).

If d is also complete then the set A can be taken to be a closed set.

Proof. The first part is a standard argument: one can construct a Lusin scheme $\langle C_s \mid s \in {}^{<\omega}\omega \rangle$ on X such that

- i) $C_\emptyset = X$
- ii) C_s is clopen
- iii) $C_s = \bigcup_{i \in \omega} C_{s \frown i}$
- iv) $\text{diam}(C_s) \leq 2^{-\text{lh}(s)}$.

From this one can conclude that the induced map f is defined on the set $A = \{y \in {}^\omega\omega \mid \bigcap_n C_{y \upharpoonright n} \neq \emptyset\}$ and is an homeomorphism. But condition *iv*) implies also $f \in \text{Lip}(A, X; 1)$. In fact, for every $x, y \in A$ such that $x \neq y$, let $n \in \omega$ be such that $d'(x, y) = 2^{-n}$ and let $s = x \upharpoonright n = y \upharpoonright n$. Clearly we have that $h(x) \in C_s$ and $h(y) \in C_s$. Thus condition *iv*) implies that $d(h(x), h(y)) \leq 2^{-\text{lh}(s)} = 2^{-n} = d'(x, y)$.

If we now assume that d is an ultrametric on X then we can construct a Lusin scheme $\langle C_s \mid s \in {}^{<\omega}\omega \rangle$ on X such that

- i) $C_\emptyset = X$
- ii) either $C_s = \emptyset$ or C_s is a sphere
- iii) $C_s = \bigcup_{i \in \omega} C_{s \frown i}$
- iv) $\text{diam}(C_s) \leq 2^{-\text{lh}(s)}$.

In fact every nonempty C_s (with $s \neq \emptyset$) will be defined as $C_s = B(x, 2^{-\text{lh}(s)})$ for some $x \in X$.

Let D be countable and dense in X : we construct the scheme by induction on $\text{lh}(s)$. First put $C_\emptyset = X$. Suppose to have constructed C_s with properties *i*)-*iv*). If $C_s = \emptyset$ then put $C_{s \frown i} = \emptyset$ for every $i \in \omega$, otherwise fix an enumeration $\langle x_k \mid k \in \omega \rangle$ of $C_s \cap D$. Then define $C_{s \frown 0} = B(x_0, 2^{-(\text{lh}(s)+1)})$ and either $C_{s \frown i+1} = B(x_{k_{i+1}}, 2^{-(\text{lh}(s)+1)})$, where k_{i+1} is the smallest $k > k_i$ such that $x_k \notin \bigcup_{j \leq i} C_{s \frown j}$, or $C_{s \frown i+1} = \emptyset$ if such a k does not exist.

Clearly $C_{s \smallfrown i} \subseteq C_s$ (since d is an ultrametric), $\text{diam}(C_{s \smallfrown i}) \leq 2^{-(\text{lh}(s)+1)}$ and $C_s = \bigcup_{i \in \omega} C_{s \smallfrown i}$ because D is dense, hence we are done. Arguing as before, h is a bijection defined on a set $A \subseteq {}^\omega \omega$ and $h \in \text{Lip}(A, X; 1)$. Now we want to show that $d'(h^{-1}(x), h^{-1}(y)) \leq 2d(x, y)$ for every distinct $x, y \in X$. Put $S_{x,y} = \{s \in {}^{<\omega} \omega \mid x \in C_s \wedge y \in C_s\}$. Clearly $S_{x,y}$ is linearly ordered and admits an element t of maximal length (otherwise $x = y$). Thus $d'(h^{-1}(x), h^{-1}(y)) = 2^{-\text{lh}(t)}$. If $d(x, y) < 2^{-(\text{lh}(t)+1)}$ then, by the construction above and the fact that d is an ultrametric, there would be an $i \in \omega$ such that $x \in C_{t \smallfrown i}$ and $y \in C_{t \smallfrown i}$, contradicting the maximality of t . Hence $d(x, y) \geq 2^{-(\text{lh}(t)+1)}$ and

$$d'(h^{-1}(x), h^{-1}(y)) = 2^{-\text{lh}(t)} = 2 \cdot 2^{-(\text{lh}(t)+1)} \leq 2d(x, y)$$

as required.

Finally it is not hard to check that the completeness of d implies that A is a closed set. \square

Note that, in particular, this Theorem provides also that every separable, metrizable and zero-dimensional space is ultrametrizable (i.e. it admits a compatible ultrametric d): let h be the homeomorphism given by the Theorem and simply put $d(x, x') = d'(h^{-1}(x), h^{-1}(x'))$ for every $x, x' \in X$, where d' is the usual (ultra)metric on ${}^\omega \omega$. Clearly, if X is also Polish we have that the ultrametric d is also complete.

For notational simplicity we put $\Sigma_0^0 = \Pi_0^0 = \Delta_0^0 = \Delta_1^0$. Moreover, for every countable ordinal ξ , we denote by $\Pi_{<\xi}^0$ (respectively, $\Sigma_{<\xi}^0$ and $\Delta_{<\xi}^0$) the pointclass $\bigcup_{\nu < \xi} \Pi_\nu^0$ (resp. $\bigcup_{\nu < \xi} \Sigma_\nu^0$ and $\bigcup_{\nu < \xi} \Delta_\nu^0$).

Theorem 4.2. *Let X be a separable, metrizable, zero-dimensional space and let A be a subset of X . For every nonzero $\xi < \omega_1$ the following are equivalent:*

- i) $A \in \Sigma_\xi^0$;
- ii) there is a Δ_ξ^0 -partition of A , i.e. there is $\langle C_n \mid n < N \rangle$ such that for every $n, m < N$ we have $C_n \in \Delta_\xi^0$, $n \neq m \Rightarrow C_n \cap C_m = \emptyset$, and $A = \bigcup_{n < N} C_n$;
- iii) there is a $\Pi_{<\xi}^0$ -partition of A , i.e. there is $\langle P_n \mid n < N \rangle$ such that for every $n < N$ there is some $\nu_n < \xi$ with $P_n \in \Pi_{\nu_n}^0$, if $n \neq m$ then $P_n \cap P_m = \emptyset$, and $A = \bigcup_{n < N} P_n$.

Proof. The implication $\text{iii}) \Rightarrow \text{ii})$ is obvious since every Π_ν^0 set is also Δ_ξ^0 if $\nu < \xi$. Also $\text{ii}) \Rightarrow \text{i})$ is easy since every Δ_ξ^0 set is by definition a Σ_ξ^0 set and the latter pointclass is closed under countable unions. Hence we have only to prove $\text{i}) \Rightarrow \text{iii})$ and this will be done by induction on $1 \leq \xi < \omega_1$.

If $\xi = 1$ we have only to note that every open set U can be written as a countable union of pairwise disjoint clopen sets. Since X is separable and zero-dimensional we have that $U = \bigcup_n C_n$ for some sets $C_n \in \Delta_1^0$: now define by induction $P_0 = C_0$ and $P_{n+1} = C_{n+1} \setminus \bigcup_{i \leq n} C_i$ and note that each P_n is clopen (since Δ_1^0 is closed under complementation and finite unions and intersections), $U = \bigcup_n P_n$ and that $n \neq m \Rightarrow P_n \cap P_m = \emptyset$.

If $\xi > 1$ and $S \in \Sigma_\xi^0$, by definition there are some sets $P_n \in \Pi_{\nu_n}^0$ such that $S = \bigcup_n P_n$ and $\nu_n < \xi$ for all $n \in \omega$. First define inductively $P'_0 = P_0$ and $P'_{n+1} = P_{n+1} \setminus \bigcup_{i \leq n} P_i$ and note that they form a partition of S . Clearly each P'_n can be seen as the difference of two Π_ν^0 sets where $\nu = \max\{\nu_0, \dots, \nu_n\} < \xi$ (since $\nu' \leq \nu \Rightarrow \Pi_{\nu'}^0 \subseteq \Pi_\nu^0$ and Π_ν^0 is closed under finite unions) and hence we have only to prove that for all $\nu < \xi$, every set of the form $Q \cap R$ with $Q \in \Pi_\nu^0$ and

$R \in \Sigma_\nu^0$ admits a Π_ν^0 partition. Using the inductive hypothesis, find a partition $\langle R_n \mid n \in \omega \rangle$ of R such that $R_n \in \Pi_{\mu_n}^0$ for some $\mu_n < \nu$ and note that $R_n \in \Pi_\nu^0$ for every $n \in \omega$. Then it is easy to check that the sets $Q_n = Q \cap R_n$ are in Π_ν^0 and that they form a partition of $Q \cap R$, hence we are done. \square

In particular, every $\Sigma_{\xi+1}^0$ set admits a Π_ξ^0 -partition. This implies that every $\Sigma_{\xi+1}^0$ -partition of X can be refined to a Π_ξ^0 -partition and, more generally, every Σ_ξ^0 -partition can be refined to a $\Pi_{<\xi}^0$ -partition. Therefore, in the case X is separable, metrizable and zero-dimensional, we have the following improvement of Theorem 3.2.

Corollary 4.3. *Let (X, d_X) and (Y, d_Y) be two metric separable spaces and assume that X is also zero-dimensional. Then a function $f : X \rightarrow Y$ is of Baire class ξ if and only if there is a sequence of functions converging uniformly to it and such that each of them is locally constant (respectively, Lipschitz, continuous) on a Π_ξ^0 -partition of X .*

5. BAIRE CLASS 1 AND FULL FUNCTIONS

Let $\Gamma \subseteq \mathcal{P}({}^\omega\omega)$ be a boldface pointclass, i.e. a collection of subsets of ${}^\omega\omega$ closed under continuous preimage. We say that a set $A \in \Gamma$ is Γ -complete if for every $B \in \Gamma$ there is a continuous function $f : {}^\omega\omega \rightarrow {}^\omega\omega$ such that $B = f^{-1}(A)$ (such a function will be called a *reduction of B in A*).

Recall also that a continuous function from ${}^\omega\omega$ to ${}^\omega\omega$ can be viewed as the function arising from some particular function $\varphi : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$. We say that $\varphi : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ is *continuous* if $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$ for every $s, t \in {}^{<\omega}\omega$ and for every $x \in {}^\omega\omega$

$$\lim_{n \in \omega} (\text{lh}(\varphi(x \upharpoonright n))) = \infty.$$

If φ is continuous it induces in a canonical way the unique function

$$f_\varphi : {}^\omega\omega \rightarrow {}^\omega\omega : x \mapsto \bigcup_{n \in \omega} \varphi(x \upharpoonright n),$$

and it is not hard to see that f_φ is a continuous function.

Conversely, suppose $f : {}^\omega\omega \rightarrow {}^\omega\omega$ is continuous. For every $s \in {}^{<\omega}\omega$ consider the set $\Sigma_s = \{t \in {}^{<\omega}\omega \mid f(\mathbf{N}_s) \subseteq \mathbf{N}_t\}$. Clearly Σ_s is linearly ordered (because if t and t' are incompatible then $\mathbf{N}_t \cap \mathbf{N}_{t'} = \emptyset$), and hence we can define $\varphi(s) = t_s$ where $t_s \in \Sigma_s$ is such that $\text{lh}(t_s) = \max\{\text{lh}(t) \mid \text{lh}(t) \leq \text{lh}(s) \wedge t \in \Sigma_s\}$. It is not difficult to check that $\varphi : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ is continuous and that $f_\varphi = f$.

By analogy with the previous definitions, if $A, B \subseteq {}^\omega\omega$ and $\varphi : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ is a continuous function such that $f_\varphi^{-1}(A) = B$, we call φ a *reduction of B into A* and we say that φ *reduces B to A* . From the observation above, it is clear that if A is Γ -complete for some pointclass $\Gamma \subseteq \mathcal{P}({}^\omega\omega)$ then for every $B \in \Gamma$ there is a reduction $\varphi : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ of B in A .

For every $t, s \in {}^{<\omega}\omega$ define $t - s = \emptyset$ if $\text{lh}(t) < \text{lh}(s)$, and $t - s = u \in {}^{<\omega}\omega$, where u is such that $t = (t \upharpoonright \text{lh}(s)) \frown u$, otherwise.

Let $\vec{\varphi} = \langle \varphi_n \mid n < N \rangle$ be a sequence of continuous functions $\varphi_n : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$. Moreover, let $\langle n_k \mid k \in \omega \rangle$ be an enumeration of N with infinite repetitions such that⁷ $n_k \neq n_{k+1}$ for every $k \in \omega$. Define $(\vec{\varphi})^* : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ and $\sigma : {}^{<\omega}\omega \rightarrow N$

⁷Clearly this last condition is required only if $N > 1$.

in the following way: first put $(\vec{\varphi})^*(\emptyset) = \emptyset$ and $\sigma(\emptyset) = n_0$. Then suppose to have defined $(\vec{\varphi})^*(s)$ and $\sigma(s) = n_k$ and inductively put

$$(\vec{\varphi})^*(s \smallfrown i) = (\vec{\varphi})^*(s) \smallfrown 1$$

if $\varphi_{\sigma(s)}(s \smallfrown i) - (\vec{\varphi})^*(s)$ does not contain 0, and

$$(\vec{\varphi})^*(s \smallfrown i) = (\vec{\varphi})^*(s) \smallfrown 0$$

otherwise. Finally put $\sigma(s \smallfrown i) = \sigma(s) = n_k$ in the first case and $\sigma(s \smallfrown i) = n_{k+1}$ in the second one.

The function $(\vec{\varphi})^*$ is clearly continuous (since it is constructed extending at each step the previous value and is such that $\text{lh}((\vec{\varphi})^*(s)) = \text{lh}(s)$ for every $s \in {}^{<\omega}\omega$) and is called the Σ_2^0 -control function⁸ of the sequence $\vec{\varphi}$, while the function σ is the *state function* associated to it. Moreover we will say that $\sigma(s) \in N$ is the *state of s with respect to $(\vec{\varphi})^*$* .

Consider now a family $A_n \subseteq {}^\omega\omega$ of Σ_2^0 sets (for $n < N$) and $S = \{x \in {}^\omega\omega \mid \exists n \forall m \geq n (x(m) \neq 0)\}$. Since S is Σ_2^0 -complete there are continuous functions $\varphi_n : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ which reduce A_n to S , i.e. such that $f_{\varphi_n}^{-1}(S) = A_n$. Define $\vec{\varphi} = \langle \varphi_n \mid n < N \rangle$ and let $(\vec{\varphi})^*$ and σ be constructed as above. For notational simplicity we put $\phi = (\vec{\varphi})^*$. We want to prove the following

Claim 5.0.1. The function $f_\phi : {}^\omega\omega \rightarrow {}^\omega\omega$ is a reduction of $\bigcup_{n < N} A_n$ in S , i.e. $f_\phi^{-1}(S) = \bigcup_{n < N} A_n$ (hence, in particular, Σ_2^0 is closed under countable unions). Moreover, $x \in \bigcup_{n < N} A_n$ if and only if the sequence $\langle \sigma(x \upharpoonright k) \mid k \in \omega \rangle$ is eventually constant, that is $\exists m \forall m' \geq m (\sigma(x \upharpoonright m') = \sigma(x \upharpoonright m))$.

Proof of the Claim. First observe that, by the definition of ϕ , for every $x \in {}^\omega\omega$ we have $f_\phi(x) \in S$ if and only if the sequence $\langle \sigma(x \upharpoonright m) \mid m \in \omega \rangle$ is eventually constant, since for every $s \in {}^{<\omega}\omega$ and $i \in \omega$ we have that $\phi(s \smallfrown i) = \phi(s) \smallfrown 0$ if and only if $\sigma(s \smallfrown i) \neq \sigma(s)$. So it is enough to prove that $x \in \bigcup_{n < N} A_n \iff \langle \sigma(x \upharpoonright m) \mid m \in \omega \rangle$ is eventually constant.

For every $k \in \omega$ let $o(n_k) = |\{i \leq k \mid n_i = n_k\}|$. Now suppose that $x \in A_n$ for some $n < N$ and let $l = |\{n \in \omega \mid f_{\varphi_n}(x)(n) = 0\}|$. Since φ_n is a reduction of A_n in S we have that $l < \omega$. Let k be such that $n_k = n$ and $o(n_k) = l + 1$. If there is no m such that $\sigma(x \upharpoonright m) = n_k$ then the sequence of states of $x \upharpoonright i$ (for $i \in \omega$) with respect to ϕ is eventually constant and we are done. Otherwise, there are $0 < m_0 < m_1 < \dots < m_{l-1} < m$ such that $\sigma(x \upharpoonright (m_i - 1)) = n$ and $\phi(x \upharpoonright m_i) - \phi(x \upharpoonright (m_i - 1))$ contains a 0 for every $i < l$. Therefore for every $m' \geq m$ we have that $\varphi_{n_k}(x \upharpoonright (m' + 1)) - \phi(x \upharpoonright m')$ does not contain any 0 since $\varphi_{n_k} = \varphi_n$: hence $\sigma(x \upharpoonright m') = n_k$ for every $m' \geq m$ and we are done again.

For the other direction, assume $x \notin \bigcup_{n < N} A_n$. Then for every $n < N$ and $m \in \omega$ we have that there is an $m' > m$ such that $\varphi_n(x \upharpoonright m') - \varphi_n(x \upharpoonright m)$ contains some 0. This implies that for every $m \in \omega$ there is an $m' > m$ such that $\sigma(x \upharpoonright m') \neq \sigma(x \upharpoonright m)$ and thus $\langle \sigma(x \upharpoonright m) \mid m \in \omega \rangle$ is not eventually constant.

□ *Claim*

With the notation above, if $x \in \bigcup_{n < N} A_n$, we will call *stabilizing point of ϕ on x* the natural number

⁸The symbol Σ_2^0 refers to the Σ_2^0 -sets which are involved in Claim 5.0.1 and in the other considerations below.

$$m_{x,\phi} = \min\{m \in \omega \mid \forall m' \geq m (\sigma(x \restriction m') = \sigma(x \restriction m))\}.$$

Moreover it is not difficult to check that in this case $x \in A_{\sigma(t)}$ (where $t = x \restriction m_{x,\phi}$). In fact, $\varphi_{\sigma(t)}(x \restriction m') - \phi(x \restriction (m' - 1))$ does not contain any 0 for every $m' \geq m_{x,\phi}$, and hence $f_{\varphi_{\sigma(t)}}(x) \in S$.

Now we state and prove a Theorem which is crucial to obtain Theorem 2.9.

Theorem 5.1. *Let A be any subset of ${}^\omega\omega$ and Y be a separable metrizable space. If $f \in \mathcal{B}_1(A, Y)$ then there is a sequence of full functions $f_k : A \rightarrow Y$ converging pointwise to f .*

Proof. Let d_Y be any compatible metric on Y . Let $\langle U_s \mid s \in {}^{<\omega}\omega \rangle$ be an *open scheme* on Y , i.e. a family of sets $U_s \subseteq Y$ such that for every $s \in {}^{<\omega}\omega$ we have⁹:

- i) $U_\emptyset = Y$
- ii) U_s is open
- iii) $U_s = \bigcup_{i \in \omega} U_{s \smallfrown i}$
- iv) $\text{diam}(U_s) \leq 2^{-\text{lh}(s)}$.

One way to do this is to fix some countable dense $D \subseteq Y$ (which exists since Y is separable) and an enumeration¹⁰ $\langle y_i \mid i \in \omega \rangle$ of it, and then recursively define $U_\emptyset = Y$ and $U_{s \smallfrown i} = B(y_i, s^{-(\text{lh}(s)+2)}) \cap U_s$. Note that U_s could be the empty set for some s , but the sequences s such that $U_s \neq \emptyset$ form a pruned tree R on ω : hence for every $s \in R$ we can fix some $y_s \in U_s$.

Since $f \in \mathcal{B}_1(A, Y)$ and $f^{-1}(U_s) \in \Sigma_2^0(A)$, there are $V_s \in \Sigma_2^0({}^\omega\omega)$ such that $f^{-1}(U_s) = V_s \cap A$ (for every $s \in R$), thus we can consider some reduction $\tilde{\varphi}_s : {}^{<\omega}\omega \rightarrow {}^{<\omega}\omega$ of V_s in $S = \{x \in {}^\omega\omega \mid \exists n \forall m \geq n (x(m) \neq 0)\} \subseteq {}^\omega\omega$. Moreover, for all these s we can consider an enumeration without repetitions $\langle j_i \mid i < I_s \rangle$ ($I_s \leq \omega$) of the $j \in \omega$ such that $s \smallfrown j \in R$, and define the sequence of continuous functions $\tilde{\varphi}_s = \langle \varphi_i \mid i < I_s \rangle$ where $\varphi_i = \tilde{\varphi}_{s \smallfrown j_i}$. Finally, let $\psi_s = (\tilde{\varphi}_s)^*$ be the Σ_2^0 -control function of the sequence $\tilde{\varphi}_s$, and σ_s be the state function associated to it.

We are now ready to define the functions f_k . Fix some $k \in \omega$ and for every $x \in A$ inductively define for $i < k$:

$$\begin{cases} s_0^{x,k} = \langle \min\{\sigma_\emptyset(x \restriction k), k\} \rangle \\ s_{i+1}^{x,k} = t_i \smallfrown \min\{\sigma_{t_i}(x \restriction k), k\}, \end{cases}$$

where $t_i = s_i^{x,k}$. For notational simplicity, for every $n \in \omega$ we will put $s_n^x = s_n^{x,n}$. Note that by definition of $\tilde{\varphi}_s$ one can easily prove by induction on $i \leq k$ that $s_k^x \in R$ for every $x \in A$: hence we can define

$$f_k : A \rightarrow Y : x \mapsto y_{s_k^x}.$$

Claim 5.1.1. For every $k \in \omega$ the function f_k is full with constant k .

Proof of the Claim. It is clear that $s_k^x \in {}^{k+1}(k+1)$ for every $x \in A$, thus f_k has at most $(k+1)^{k+1}$ values. Moreover, these values depend only on the sequence $x \restriction k$, hence the preimage of each of them is a union of balls with radius 2^{-k} (and hence is a full set by Proposition 2.7). \square *Claim*

⁹Note that in general this is not a Lusin scheme since, in this case, we do not require that if s and t are incompatible sequences then $U_s \cap U_t = \emptyset$. However we can add this condition if Y is also zero-dimensional.

¹⁰We allow repetitions if Y is finite.

Claim 5.1.2. The sequence $\langle f_k \mid k \in \omega \rangle$ converges to f pointwise.

Proof of the Claim. Fix some $x \in A$ and $n \in \omega$. We want to prove that there is an $m \in \omega$ such that $\forall m' \geq m$ ($d_Y(f_{m'}(x), f(x)) \leq 2^{-(n+1)}$).

We define inductively a sequence $\langle m_j \mid j \leq n \rangle$ of natural numbers (the sequence of the stabilizing points of x) and a sequence $\langle t_j \mid j \leq n \rangle$ of compatible and length increasing sequences of natural numbers. First put $m_0 = m_{x, \psi_\emptyset}$ and $t_0 = \langle \sigma_\emptyset(x \upharpoonright m_0) \rangle$: then for every $i < n$ define $m_{i+1} = m_{x, \psi_{t_i}}$ and $t_{i+1} = t_i \hat{\ } \sigma_{t_i}(x \upharpoonright m_{i+1})$. Finally put $t_{-1} = \emptyset$ by definition.

Now recall that, by Claim 5.0.1 and the observations following it, if $f(x) \in \bigcup_{m \in \omega} U_{s \hat{\ } m}$ then $f(x) \in U_{s \hat{\ } \sigma_s(x \upharpoonright m)}$ for every $m \geq m_{x, \psi_s}$. Therefore the fact that $f(x) \in Y$ implies that $f(x) \in U_{t_0}$. Moreover, using the same argument, one can show that since $f(x) \in U_{t_i}$ then $f(x) \in U_{t_{i+1}}$ for every $i < n$, hence we have $f(x) \in U_{t_n}$.

Recall also that, by definition of the numbers m_i ,

$$\forall m' \geq m_i \ (t_i = t_{i-1} \hat{\ } \sigma_{t_{i-1}}(x \upharpoonright m')).$$

Let $m = \max\{m_0, \dots, m_n, n, k\}$, where k is the smallest natural number such that $t_n \in {}^{<\omega}k$. Again by induction on $i \leq n$, it is not hard to prove that for every $m' \geq m$ and every $i \leq n$ we have $s_i^{x, m'} = t_i$ and hence $s_{m'}^x \supseteq t_n$. Since we have that $f_{m'}(x) = y_{s_{m'}^x} \in U_{s_{m'}^x} \subseteq U_{t_n}$ (by $s_{m'}^x \supseteq t_n$), $\text{lh}(t_n) = n + 1$, and $\text{diam}(U_{t_n}) = 2^{-\text{lh}(t_n)}$, we can conclude that $d_Y(f_{m'}(x), f(x)) \leq 2^{-(n+1)}$ and we are done. \square *Claim*

\square

We are now ready to prove the characterization of the Baire class 1 functions as pointwise limits of full functions, i.e. Theorem 2.9.

Proof of Theorem 2.9. Since every full function is Lipschitz and every Lipschitz function is continuous, if f is the pointwise limit of a sequence of full functions then it is in $\mathcal{B}_1(X, Y)$ (see note 2 on page 3).

For the other direction, let X and Y be as in the hypotheses of the Theorem, and let $A \subseteq {}^\omega\omega$ and $h : A \rightarrow X$ be obtained applying the second part of Theorem 4.1 to X . Define

$$g = f \circ h : A \rightarrow Y.$$

Since h is continuous, if f is of Baire class 1 then also g is of Baire class 1. Let $g_n : A \rightarrow Y$ be the sequence of full functions convergent (pointwise) to g that comes from Theorem 5.1, and define for every $n \in \omega$

$$f_n = g_n \circ h^{-1} : X \rightarrow Y.$$

Clearly each f_n is a full function by Proposition 2.8, and moreover f is the pointwise limit of the sequence $\langle f_n \mid n \in \omega \rangle$: in fact for every $x \in X$ and every $n \in \omega$ we have that $d(f_m(x), f(x)) = d(g_m(h^{-1}(x)), g(h^{-1}(x))) \leq 2^{-n}$ for m large enough (since $g_m \rightarrow g$ pointwise). This completes the proof. \square

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KURT GÖDEL RESEARCH CENTER FO MATHEMATICAL LOGIC, UNIVERSITY OF VIENNA, AUSTRIA
E-mail address: `luca.mottoros@libero.it`